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Formula for the reflection length of elements in the group $G(m, p, n)$ ☆

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Abstract

We deduce a closed formula for the reflection length functions on the reflection group $G(m, p, n)$ with $p \in [m]$ and $p \mid m$. The formula is shown separately in three cases: $p = 1$, $p = m$ and $p \in [2, m - 1]$ (see the proof of Theorems 2.1, 3.1 and 4.4).

Keywords: Reflections; Reflection groups; Reflection length

0. Introduction

0.1. Let $\mathbb{P}$ (respectively, $\mathbb{N}$, $\mathbb{Z}$) denote the set of positive integers (respectively, non-negative integers, integers). For any $k < n$ in $\mathbb{N}$, let $[k, n] := \{k, k + 1, \dots, n\}$ and $[n] := [1, n]$. Fix $m, p, n \in \mathbb{P}$ with $p \mid m$ (reading "p divides m"), let $G(m, p, n)$ be the reflection group consisting of all the $n \times n$ monomial matrices w such that all the non-zero entries, say $\theta_1, \dots, \theta_n$, of w are mth roots of unity with $(\prod_{i=1}^{n} \theta_i)^{m/p} = 1$. Any $w \in G(m, p, n)$ can be expressed in the form $w = [a_1, \dots, a_n \mid \sigma]$ with $\sigma \in S_n$, where S_n is the symmetric group on the set $[n]$, and the entry in the $(k, (k)\sigma)$-position of w is $\exp(\frac{2\pi a_k \sqrt{-1}}{m})$ for $k \in [n]$ (note that the notation

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$[a_1, \ldots, a_n \mid \sigma]$ given here is just the notation $[\exp(\frac{2\pi a_1\sqrt{-1}}{m}), \ldots, \exp(\frac{2\pi a_n\sqrt{-1}}{m}) \mid \sigma]$ given in [5,6]). By the condition on w, we have $p \mid \sum_{k=1}^{n} a_k$.

We assume $p \mid m$ throughout the paper whenever $G(m, p, n)$ is mentioned.

0.2. A linear transformation of a complex vector space V is called a *reflection* (sometimes called a *pseudo-reflection* in the literature to distinguish it from the concept of a reflection in a euclidean space), if it is of finite order whose fixed point space is a hyperplane of V. A group G generated by reflections is called a *reflection group*. Let T be the set of all the reflections in G. Any $w \in G$ can be expressed in the form $w = s_1 s_2 \cdots s_r$ with $s_i \in T$. Denote by $l_T(w)$ the smallest possible number r of factors among all such expressions, call $l_T(w)$ the *reflection length* of w. For any $w \in G$, let $V^w := \{v \in V \mid (v)w = v\}$. Then the inequality $l_T(w) \geqslant \operatorname{codim}_V V^w$ holds in general.

0.3. Under its natural action on the space $V = \mathbb{C}^n$, an element $w = [a_1, \ldots, a_n \mid \sigma]$ of $G(m, p, n)$ is a reflection if and only if one of the following conditions holds:

(1) $\sigma = (i, j)$ is a transposition of i and j for some $i < j$ in $[n]$, $a_j \equiv -a_i \pmod m$, and $a_k \equiv 0 \pmod m$ for $k \neq i, j$. Denote w by $t(i, j; a_i)$ or $t(j, i; -a_i)$, and call it *a reflection of type I*.
(2) $w \neq 1$ is diagonal with $n - 1$ diagonal entries being 1 (such kind of reflections exists only when $p < m$). Denote w by $s(k; a_k)$ and call it *a reflection of type II* if $a_k \not\equiv 0 \pmod m$.

0.4. Note that the group $G(m, p, n)$ has a well-known presentation whose generating reflection set S consists of

(i) $n + 1$ reflections: t_1', s, and t_i, $i \in [n-1]$, if $p \in [2, m-1]$;
(ii) n reflections: s and t_i, $i \in [n-1]$, if $p = 1$;
(iii) n reflections: t_n' and t_i, $i \in [n-1]$, if $p = m$,

where $t_i = t(i, i+1; 0)$, $t_1' = t(1, 2; -1)$, $t_n' = t(1, n; -1)$ and $s = s(1; p)$ (see [1, Appendix II] and [4, Proposition 3.3]). The *length function* $l_S(w)$ of w is defined to be the smallest possible number r of factors such that there exists an expression $w = s_1 s_2 \cdots s_r$ with $s_i \in S$.

0.5. The group $G(m, p, n)$ has many other presentations and hence many different length functions accordingly (see [4–6]). The function $l_T(w)$ on $G(m, p, n)$ is presentation-free, satisfying $l_T(w) \leqslant l_S(w)$ for any generating reflection set S of $G(m, p, n)$.

Except for the cases of the groups $G(m, 1, n)$, $G(m, m, n)$ with the generator sets S as in 0.4(ii)–(iii) (see [2] and [4, Propositions 2.5 and 3.5]), so far there is no closed formula for the function $l_S(w)$ for the group $G(m, p, n)$ with S in all the other cases.

0.6. In the present paper, we shall deduce a closed formula of the function $l_T(w)$ for any $w \in G(m, p, n)$. We consider three cases $p = 1$ $p = m$ and $p \in [2, m-1]$ separately. The results are included in Theorems 2.1, 3.1 and 4.4. It should be emphasised that the formula in Theorem 4.4 simultaneously covers all the three cases for p. We also give a necessary and sufficient condition on the validity of the equation $l_T(w) = \operatorname{codim}_V V^w$ for $w \in G(m, p, n)$ (see Remarks 2.3(1), 3.4(3) and Proposition 5.3(2)).

The formulae for the reflection lengths can be used to study a partial ordering, called *reflection ordering*, on the elements of $G(m,p,n)$, the latter can be expected to play an important role in the theory of the groups $G(m,p,n)$, just as the Bruhat–Chevalley order in the theory of the Coxeter groups. The reflection ordering on $G(m,p,n)$ will be studied in a forthcoming paper.

0.7. The contents of the paper are organised as follows. Section 1 contains the preliminaries, we collect some concepts and results for subsequent use. Then in Sections 2–4, we prove the formulae of $l_T(w)$ on $G(m,p,n)$ in the cases of $p=1$, $p=m$ and $p\in[2,m-1]$ separately, one section for each case. In Section 5, we study some properties of $l_T(w)$ by applying Theorem 4.4.

1. Preliminaries

We collect some concepts and results in this section for later use.

Lemma 1.1. *Let $w=[a_1,\dots,a_n\mid(1,2,\dots,r)]\in G(m,p,n)$ and $s=t(1,j;a)$ for some $j,r\in[n]$ with $j\in[2,r-1]$ and $a_i,a\in\mathbb{Z}$. Then*

$$\begin{aligned} ws=\bigl[&a_1,\dots,a_{j-2},a_{j-1}-a,a_j,\dots,a_{r-1},\\ &a_r+a,a_{r+1},\dots,a_n\bigm|(1,2,\dots,j-1)(j,j+1,\dots,r)\bigr],\end{aligned}$$

where the notation $(i_1,i_2,\dots,i_k)$ stands for the cyclic permutation on $[n]: i_j\mapsto i_{j+1}$ for $j\in[k-1]$, $i_k\mapsto i_1$, and $h\mapsto h$ for $h\notin\{i_j\mid j\in[k]\}$.

Proof. This can be shown by direct calculation. □

Lemma 1.2. *Let $w'=[a'_1,\dots,a'_n\mid\sigma']\in G(m,p,n)$, where σ' is the cyclic permutation $(1,2,\dots,r)$ for some $r\in[n]$. Given $r_1<r_2<\dots<r_k=r$ in $[n]$ and integers $b_1,b_2,\dots,b_{k-1}$, let $w=w'\cdot t(1,r_1+1;b_1)t(r_1+1,r_2+1;b_2)\cdots t(r_{k-2}+1,r_{k-1}+1;b_{k-1})$. Then $w=[a_1,a_2,\dots,a_n\mid\sigma]$ with $\sigma=(1,2,\dots,r_1)(r_1+1,r_1+2,\dots,r_2)\ \cdots\ (r_{k-1}+1,r_{k-1}+2,\dots,r_k)$, where*

$$a_i=\begin{cases} a'_i, & \text{if } i\notin\{r_1,\dots,r_k\},\\ a'_{r_j}-b_j, & \text{if } i=r_j \text{ for some } j\in[k-1],\\ a'_{r_k}+\sum_{j\in[k-1]}b_j, & \text{if } i=r_k.\end{cases}\tag{1.2.1}$$

Proof. This follows by repeatedly applying Lemma 1.1. □

Remark 1.3. Let $\sigma,\sigma'\in S_n$ be as in Lemma 1.2. Suppose that $w'\in G(m,p,n)$ in Lemma 1.2 is given. Then for any $I\subset[k]$ with $|I|=k-1$ and for any $(c_j)_{j\in I}\in\mathbb{Z}^{|I|}$, there exists a unique sequence of integers $b_1,b_2,\dots,b_{k-1}$ such that $w=w'\cdot t(1,r_1+1;b_1)t(r_1+1,r_2+1;b_2)\cdots t(r_{k-2}+1,r_{k-1}+1;b_{k-1})=[a_1,\dots,a_n\mid\sigma]$, where $a_{r_j}=c_j$ for any $j\in I$. On the other hand, suppose that the element w in Lemma 1.2 is given. Then for any $I\subset[k]$ with $|I|=k-1$ and for any $(c_j)_{j\in I}\in\mathbb{Z}^{|I|}$, there exists a unique sequence of integers $b_1,b_2,\dots,b_{k-1}$ such that $w'=w\cdot t(1,r_1+1;b_1)t(1,r_2+1;b_2)\cdots t(1,r_{k-1}+1;b_{k-1})=[a'_1,\dots,a'_n\mid\sigma']$, where $a'_{r_i}=c_i$ for any $i\in I$.

These facts will be very useful in the later sections.

1.4. By *a partition* $E = \{E_1, \ldots, E_l\}$ (respectively, $E = [\![E_1, \ldots, E_l]\!]$) of a set (respectively, a multi-set) X, we mean that $X = \bigcup_{i \in [l]} E_i$ is a disjoint union of non-empty subsets (respectively, sub-multi-sets) $E_1, \ldots, E_l$. We call E_i a *block* of E.

For $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, p, n)$, let $P = \{P_1, \ldots, P_t\}$ be a partition of $[n]$ such that $p \mid \sum_{i \in P_j} a_i$ for any $j \in [t]$, and each P_j is σ-stable (i.e., P_j is a union of some σ-orbits). Define $w_k = [a_{k1}, \ldots, a_{kn} \mid \sigma_k] \in G(m, p, n)$, $k \in [t]$, by

$$a_{kj} = \begin{cases} a_j, & \text{if } j \in P_k, \\ 0, & \text{otherwise}, \end{cases} \qquad (h)\sigma_k = \begin{cases} (h)\sigma, & \text{if } h \in P_k, \\ h, & \text{otherwise}. \end{cases}$$

Then we have $w_j w_k = w_k w_j$ for any $h, k \in [t]$ and $w = w_1 w_2 \cdots w_t$. For any $k \in [t]$, let $n_k = |P_k|$. Then there exists a unique bijective map $\phi_k : P_k \to [n_k]$ satisfying that $i < j$ in P_k implies $\phi_k(i) < \phi_k(j)$. ϕ_k determines a unique injective group homomorphism $\psi_k : G(m, p, n_k) \to G(m, p, n)$ such that for any $y = [b_1, \ldots, b_{n_k} \mid \tau_k] \in G(m, p, n_k)$, the element $\psi_k(y) = [b_{k1}, \ldots, b_{kn} \mid \tau] \in G(m, p, n)$ is given by

$$b_{kj} = \begin{cases} b_{\phi_k(j)}, & \text{if } j \in P_k, \\ 0, & \text{otherwise}, \end{cases} \qquad (h)\tau = \begin{cases} \phi_k^{-1}((\phi_k(h))\tau_k), & \text{if } h \in P_k, \\ h, & \text{otherwise}. \end{cases}$$

We see that ψ_k sends reflections of $G(m, p, n_k)$ to reflections of $G(m, p, n)$.

1.5. Given $m, p, r \in \mathbb{P}$ with $p \mid m$. Let $C = [\![c_1, c_2, \ldots, c_r]\!]$ be a multi-set of r integers.

A subset E of $[r]$ is called (C, m)-*perfect* if $\sum_{h \in E} c_h \equiv 0 \pmod{m}$. A partition $P = \{P_1, \ldots, P_l\}$ of $[r]$ is called (C, m)-*admissible* if P_j is (C, m)-perfect for any $j \in [l]$. Let $\Lambda(C; m)$ be the set of all the (C, m)-admissible partitions of $[r]$. We see that $\Lambda(C; m) \neq \emptyset$ if and only if $\sum_{h \in [r]} c_h \equiv 0 \pmod{m}$. When $\Lambda(C; m) \neq \emptyset$, denote by $t(P)$ the number of blocks of P for any $P \in \Lambda(C; m)$, and define $t(C, m) = \max\{t(P) \mid P \in \Lambda(C; m)\}$.

A subset E of $[r]$ is called (C, m, p)-*semi-perfect*, if $\sum_{h \in E} c_h \equiv 0 \pmod{p}$ and $\sum_{h \in E} c_h \not\equiv 0 \pmod{m}$. Such a subset E of $[r]$ possibly exists only if $p < m$. A partition $P = \{P_1, \ldots, P_l\}$ of $[r]$ is (C, m, p)-*semi-admissible* if P_j is either (C, m)-perfect or (C, m, p)-semi-perfect for any $j \in [l]$. Let $\Lambda(C; m, p)$ be the set of all the (C, m, p)-semi-admissible partitions of $[r]$. We have $\Lambda(C; m, p) \neq \emptyset$ if and only if $\sum_{h \in [r]} c_h \equiv 0 \pmod{p}$. When $\Lambda(C; m, p) \neq \emptyset$, denote by $t(P)$ (respectively, $u(P)$) the number of (C, m)-perfect (respectively, (C, m, p)-semi-perfect) blocks of P and define $v(P) = 2t(P) + u(P)$ for any $P \in \Lambda(C; m, p)$. Define $v(C, m, p) = \max\{v(P) \mid P \in \Lambda(C; m, p)\}$ if $\Lambda(C; m, p) \neq \emptyset$. We have $\Lambda(C; m, m) = \Lambda(C; m)$ and $v(C, m, m) = 2t(C, m)$ if $\Lambda(C; m) \neq \emptyset$.

Lemma 1.6. *Let $C = [\![c_1, \ldots, c_r]\!]$ (respectively, $C' = [\![c'_1, \ldots, c'_{r+1}]\!]$) be a multi-set of r (respectively, $r + 1$) integers with $c'_r + c'_{r+1} = c_r$ and $c'_i = c_i$ for any $i \in [r - 1]$. Then the inequalities $t(C, m) \leqslant t(C', m) \leqslant t(C, m) + 1$ hold, provided that $\Lambda(C; m) \neq \emptyset$.*

Proof. Set $t = t(C, m)$ and $t' = t(C', m)$.

(i) There exists some $P = \{P_1, \ldots, P_t\}$ in $\Lambda(C; m)$. Suppose that $r \in P_j$ for some $j \in [t]$. Let $P'_j = P_j \cup \{r + 1\}$. Then $P' = \{P_1, \ldots, P_{j-1}, P'_j, P_{j+1}, \ldots, P_t\}$ is in $\Lambda(C'; m)$. This implies $t \leqslant t'$.

(ii) There exists some $P' = \{P'_1, \ldots, P'_{t'}\}$ in $\Lambda(C'; m)$. Let $r \in P'_i$ and $r + 1 \in P'_j$ for some $i, j \in [t']$. If $i = j$, then $P = \{P'_1, \ldots, P'_{i-1}, P'_i \setminus \{r + 1\}, P'_{i+1}, \ldots, P'_{t'}\}$ is in $\Lambda(C; m)$

and hence $t = t'$ by (i). If $i \neq j$, we may assume $i < j$ for the sake of definiteness, then $P = \{P_1', \ldots, \widehat{P_i'}, \ldots, \widehat{P_j'}, \ldots, P_{t'}', (P_i' \cup P_j') \setminus \{r+1\}\}$ is in $\Lambda(C; m)$ and hence $t \geqslant t' - 1$, where the notation $\widehat{P_h'}$ stands for the deletion of the term P_h'.

So our result follows from (i)–(ii). $\square$

1.7. For any $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, p, n)$, write σ as a product of disjoint cyclic permutations:

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_r, \tag{1.7.1}$$

where $\sigma_k = (i_{k1}, i_{k2}, \ldots, i_{km_k})$ for $k \in [r]$ with $\sum_{j \in [r]} m_j = n$. Set $r(w) = r$ and $|\sigma_k| = m_k$. Let $I_j = \{i_{j1}, i_{j2}, \ldots, i_{jm_j}\}$ for $j \in [r]$. Then $I(w) = \{I_1, \ldots, I_r\}$ is a partition of $[n]$ determined by w. Let $c_j = \sum_{k \in I_j} a_k$ and let $C(w) = [\![c_1, c_2, \ldots, c_r]\!]$. Set $\Lambda(w; m, p) := \Lambda(C(w); m, p)$.

When $w \in G(m, m, n)$, we have $\Lambda(w; m, m) = \Lambda(C(w); m)$ since $\Lambda(C(w); m, m) = \Lambda(C(w); m)$.

For $w \in G(m, p, n)$, we always have $\Lambda(w; m, p) \neq \emptyset$ since $[r]$ is in $\Lambda(w; m, p)$ as a partition of itself.

Set $t(w) := t(C(w), m)$ if $p = m$ and $v(w) = v(C(w), m, p)$ if $p \in [m]$.

We always have $t(w) > 0$ and $v(w) > 0$ whenever it is applicable. In particular, $v(w) = 2t(w)$ if $w \in G(m, m, n)$.

Let $t_0(w) = \#\{j \in [r] \mid c_j \equiv 0 \pmod m\}$. Then $t_0(w) = \operatorname{codim}_V V^w$ (see 0.2). In particular, when $w \in G(m, 1, n)$, we have $v(w) = t_0(w) + r(w)$.

Corollary 1.8. *In the setup of* 1.7 *with* $w = [a_1, \ldots, a_n \mid \sigma] \in G(m, p, n)$*, let* $s = t(i, j; a)$ *be with* $i, j \in I_k$ *for some* $k \in [r]$*. Then* $C(ws) = [\![c_1, \ldots, \widehat{c_k}, \ldots, c_r, c, d]\!]$ *for some* $c, d \in \mathbb{Z}$ *with* $c + d = c_k$*.*

Proof. This follows by Lemma 1.1. $\square$

We record the following simple results which will be used implicitly in the subsequent sections.

Lemma 1.9. *For any* $i \neq j$*,* $h \neq k$ *in* $[n]$ *and* $a, b \in \mathbb{Z}$*, we have*

$$t(i, j; a)t(h, k; b) = \begin{cases} t(h, k; b)t(i, j; a), & \text{if } \{i, j\} \cap \{h, k\} = \emptyset, \\ t(h, k; b)t(k, j; a - b), & \text{if } i = h \text{ and } j \neq k, \\ t(h, k; b)t(i, h; a - b), & \text{if } i \neq h \text{ and } j = k, \\ s(i; a - b)s(j; b - a), & \text{if } i = h \text{ and } j = k. \end{cases}$$

2. The group $G(m, 1, n)$

In this section, we prove a formula for the function $l_T(w)$ on the group $G(m, 1, n)$.

Theorem 2.1. $l_T(w) = n - t_0(w)$ *for any* $w \in G(m, 1, n)$*.*

Proof. Keep the setup of 1.7 with $w = [a_1, \dots, a_n \mid \sigma]$ and $r = r(w)$. We first show the inequality

$$l_T(w) \leqslant n - t_0(w). \tag{2.1.1}$$

Let $t = t_0(w)$. We may assume $c_j \equiv 0 \pmod m$ for $j \in [t]$ and $c_l \not\equiv 0 \pmod m$ for $l \in [t+1, r]$ by relabelling the c_h's if necessary. Denote $t_{h,j} = t(i_{h,j}, i_{h,j+1}; a_{i_{h,j}})$ and $s_k = s(i_{k,m_k}; c_k)$ for any $h \in [r]$, $j \in [m_h - 1]$ and $k \in [t+1, r]$. Then we have

$$w = \prod_{h=1}^{t} (t_{h,m_h-1} t_{h,m_h-2} \cdots t_{h,1}) \cdot \prod_{j=t+1}^{r} (s_j t_{j,m_j-1} t_{j,m_j-2} \cdots t_{j,1}).$$

This implies that

$$l_T(w) \leqslant \sum_{h=1}^{t} (m_h - 1) + \sum_{j=t+1}^{r} m_j = \sum_{h=1}^{r} m_h - t = n - t_0(w)$$

which proves (2.1.1).

We claim that $t_0(ws) \leqslant t_0(w) + 1$ for any reflection s of $G(m, 1, n)$. First let $s = t(i, j; a)$ for some $i \neq j$ in $[n]$ and $a \in \mathbb{Z}$. If $i, j \in I_h$ for some $h \in [r]$, then by Corollary 1.8, we have $C(ws) = [\![c_1, \dots, \widehat{c_h}, \dots, c_r, c, d]\!]$ for some $c, d \in \mathbb{Z}$ with $c + d = c_h$. If $i \in I_h$ and $j \in I_k$ for some $h \neq k$ in $[r]$, then $C(ws) = [\![c_1, \dots, \widehat{c_h}, \dots, \widehat{c_k}, \dots, c_r, c_h + c_k]\!]$ again by Corollary 1.8. Next assume $s = s(i; a)$ for some $i \in [n]$ and $a \in \mathbb{Z}$ with $m \nmid a$. Then $i \in I_k$ for some $k \in [r]$. The multi-set $C(ws)$ is $[\![c_1, \dots, c_{k-1}, c_k + a, c_{k+1}, \dots, c_r]\!]$. Concerning the above three cases, the only possibility for violating the inequality $t_0(ws) \leqslant t_0(w) + 1$ is in the first case, where $c \equiv 0 \equiv d \pmod m$ and $c_h = c + d \not\equiv 0 \pmod m$. But this is impossible. So we always have $t_0(ws) \leqslant t_0(w) + 1$.

Write $w = s_1 s_2 \cdots s_l$ with $l = l_T(w)$. Then by repeatedly applying the above claim, we have $t_0(w) \geqslant t_0(1) - l = n - l_T(w)$, i.e., $l_T(w) \geqslant n - t_0(w)$. This completes our proof. $\square$

Example 2.2. Let $w = [2, 1, 7, 5, 2, 6, 2, 3 \mid (1)(23)(457)(68)] \in G(9, 1, 8)$. Then w determines a partition $\{\{1\}, \{2, 3\}, \{4, 5, 7\}, \{6, 8\}\}$ of the set $[8]$ and hence a partition $[\![\, [\![2]\!], [\![1, 7]\!], [\![5, 2, 2]\!], [\![6, 3]\!] \,]\!]$ of the multi-set $[\![2, 1, 7, 5, 2, 6, 2, 3]\!]$. So $C(w) = [\![c_1, \dots, c_4]\!] = [\![2, 8, 9, 9]\!]$. There are just two numbers, i.e., $9, 9$, in $C(w)$ are multiples of 9. Hence $t_0(w) = 2$. We get $l_T(w) = 8 - 2 = 6$.

Remark 2.3.

(1) By Theorem 2.1 and the fact $t_0(w) = \dim V^w$, we have $l_T(w) = \operatorname{codim}_V V^w$ for any $w \in G(m, 1, n)$ (see 0.2). The function $l_T(w)$ reaches its maximal value n if and only if $t_0(w) = 0$, i.e., none of the numbers c_j, $j \in [r(w)]$, contained in $C(w)$ is divisible by m.

(2) We have the isomorphisms $G(1, 1, n) \cong A_{n-1} = S_n$ and $G(2, 1, n) \cong B_n$, where A_{n-1}, B_n are the Weyl groups W of types A_{n-1}, B_n, respectively. In [3], Carter showed that any element of a Weyl group W of rank n can be written as a product of $\leqslant n$ reflections. Theorem 2.1 gives a more precise result on the function $l_T(w)$ for the groups A_{n-1}, B_n.

3. The group $G(m, m, n)$

We prove a formula for the function $l_T(w)$ on the group $G(m, m, n)$ in this section.

Recall the notation $t(w) = t(C(w), m)$ and $\Lambda(w; m, m) = \Lambda(C(w); m)$ for any $w \in G(m, m, n)$ in 1.7.

Theorem 3.1. *$l_T(w) = n + r(w) - 2t(w)$ for any $w \in G(m, m, n)$.*

Proof. Let $w = [a_1, \dots, a_n \mid \sigma]$ be with $r = r(w)$.

(1) First show the inequality

$$l_T(w) \leqslant n + r - 2t(w). \tag{3.1.1}$$

(1a) First assume $t(w) = 1$. We may assume without loss of generality that $\sigma = (1, 2, \dots, k_1) \times (k_1 + 1, \dots, k_2) \cdots (k_{r-1} + 1, \dots, k_r)$ for some $1 \leqslant k_1 < k_2 < \cdots < k_r = n$. Let $w' = w \cdot t(1, k_1+1; 0)t(1, k_2+1; 0) \cdots t(1, k_{r-1}+1; 0)$. Then $w' = [a_1, \dots, a_n \mid \sigma']$ with σ' the cyclic permutation $(1, 2, \dots, n)$ by Lemma 1.2. Hence $w't(1, 2; a_1)t(2, 3; a_2) \cdots t(n-1, n; a_{n-1}) = 1$. This implies $l_T(w) \leqslant n + r - 2$.

(1b) Keep the notation in 1.7 for w. In particular, write σ as in (1.7.1). Next assume that $P = \{P_1, \dots, P_t\}$ is a partition of $[r]$ in $\Lambda(w; m, m)$ with $t = t(w)$ and $|P_h| = p_h$ for $h \in [t]$. Hence $\sum_{h\in[t]} p_h = r$. For $h \in [r]$ and $k \in [t]$, let $m_h = |I_h|$ and $n_k = \sum_{h\in P_k} m_h$ (see 1.7 for the notation I_h and m_h). Then $\sum_{h\in[t]} n_h = n$. For any $k \in [t]$, define $w_k = [a_{k1}, \dots, a_{kn} \mid \sigma_k]$ by

$$a_{kj} = \begin{cases} a_j, & \text{if } j \in I_h \text{ for some } h \in P_k, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_k = \prod_{h\in P_k} (i_{h1}, i_{h2}, \dots, i_{hm_h}).$$

By 1.4, we see that w_k can be identified with an element, say x_k, in $G(m, m, n_k)$ with $r(x_k) = p_k$ and $t(x_k) = 1$. Hence by (1a), x_k can be written as a product of $n_k + p_k - 2$ reflections in $G(m, m, n_k)$. Thus w_k is also in $G(m, m, n)$.

We have $w = \prod_{k\in[t]} w_k$. So w can be written as a product of $\sum_{h\in[t]}(n_h + p_h - 2) = n + r(w) - 2t(w)$ reflections. This implies the inequality (3.1.1).

(2) Next show that the inequality

$$n + r(w) - 2t(w) \leqslant n + r(ws) - 2t(ws) + 1 \tag{3.1.2}$$

holds for any reflection s in $G(m, m, n)$. We can write $s = t(h, k; a)$ for some $h < k$ in $[n]$ and $a \in \mathbb{Z}$. Let $I = \{I_1, \dots, I_r\}$ be the partition of the set $[n]$ determined by w and let $C(w) = [\![c_1, \dots, c_r]\!]$ be with $c_j = \sum_{h\in I_j} a_h$ for $j \in [r]$. If $h, k \in I_j$ for some $j \in [r]$ then $C(wx) = [\![c_1, \dots, \widehat{c_j}, \dots, c_r, c, d]\!]$ for some $c, d \in \mathbb{Z}$ with $c + d = c_j$ by Corollary 1.8; if $h \in I_j$ and $k \in I_l$ with $j \neq l$ in $[r]$ (we may assume $j < l$ for the sake of definiteness), then $C(ws) = [\![c_1, \dots, \widehat{c_j}, \dots, \widehat{c_l}, \dots, c_r, c_j + c_l]\!]$ again by Corollary 1.8. By Lemma 1.6, we have $r(ws) = r(w) + 1$ and $t(ws) \leqslant t(w) + 1$ in the former case; and $r(ws) = r(w) - 1$, $t(ws) \leqslant t(w)$ in the latter case. Hence the inequality (3.1.2) holds in either case.

(3) Write $w = s_1 s_2 \cdots s_l$ with $l = l_T(w)$ for some reflections $s_1, \dots, s_l$ in $G(m, m, n)$. By repeatedly applying (3.1.2), we get

$$n + r(w) - 2t(w) \leqslant n + r(1) - 2t(1) + l = l_T(w) \tag{3.1.3}$$

since $r(1) = t(1) = n$. Therefore our result follows by (3.1.1) and (3.1.3). $\square$

We have $G(2,2,n) \cong D_n$, the Weyl group of type D_n. Then the following result shows that Theorem 3.1 is compatible with the Carter's result in [3] that every element of a Weyl group W of rank n can be expressed as a product of $\leqslant n$ reflections in W. The following gives a more precise result on the function $l_T(w)$ for the group D_n.

Corollary 3.2. *Let $w \in G(2,2,n)$ be with $r = r(w)$. Keeping the notation in* 1.7 *for w, we have*

(1) $t(w) \geqslant \frac{1}{2}r(w)$.
(2) $t(w) = \frac{1}{2}r(w)$ *if and only if* $t_0(w) = 0$. *When the equivalent conditions hold, the value $r(w)$ is even and $c_j \equiv 1 \pmod 2$ for any $j \in [r]$.*
(3) $l_T(w) = n - t_0(w)$. *In particular, $l_T(w) \leqslant n$.*

Proof. The results follow easily by Theorem 3.1 and by the facts that $t_0(w) \equiv r(w) \pmod 2$ and $t(w) = t_0(w) + \frac{1}{2}(r(w) - t_0(w))$. □

Example 3.3. Let $w = [2,4,8,5,2,6,2,7 \mid (158)(47)(236)] \in G(9,9,8)$. Then $r(w) = 3$ and w determines a partition $I = \{I_1, I_2, I_3\}$ of the set [8] with $I_1 = \{1,5,8\}$, $I_2 = \{4,7\}$ and $I_3 = \{2,3,6\}$. We also have $C(w) = [\![c_1, c_2, c_3]\!]$ with $c_1 = 11$, $c_2 = 7$ and $c_3 = 18$. Let $P = \{P_1, P_2\}$ be with $P_1 = \{1,2\}$ and $P_2 = \{3\}$. Then P is a $(C(w), 9)$-admissible partition of [3]. We have $t(w) = 2$. Hence $l_T(w) = 8 + 3 - 4 = 7$.

Remark 3.4.

(1) By Corollary 3.2, we see that the formula of $l_T(w)$ for $G(2,2,n)$ coincides with that for $G(m,1,n)$.
(2) Let $w \in G(m,m,n)$. When $r(w) = 1$, we have $t(w) = 1$ by the fact that $t(w) \in [r(w)]$. Hence $l_T(w) = n - 1$. When w is a permutation matrix, we have $t(w) = r(w)$ and hence $l_T(w) = n - r(w)$. When w is diagonal, we have $r(w) = n$. In this case, if $t(w) = 1$, i.e., no sum of any $k < n$ of its diagonal entries is divisible by m, then $l_T(w)$ reaches its maximal value $2n - 2$. Conversely, if $w \in G(m,m,n)$ satisfies $l_T(w) = 2n - 2$, then w must be a diagonal matrix.
(3) By Theorem 3.1 and the fact $t_0(w) = \dim V^w$, we see that for any $w \in G(m,m,n)$, the equation $l_T(w) = \mathrm{codim}_V V^w$ holds if and only if there exists some $P = (P_1, P_2, \ldots, P_t) \in \Lambda(w; m, m)$ with $|P_i| \leqslant 2$ for any $i \in [t]$ (see 0.2 and 1.5). In particular, the equivalent conditions always hold in the group $G(2,2,n)$ by Corollary 3.2(3).

4. The groups $G(m, p, n)$

In the present section, we shall extend Theorems 2.1 and 3.1 to the more general groups $G(m,p,n)$ with $p \in [m]$ and $p \mid m$. The main result of the section is Theorem 4.4.

We always assume $p \mid m$ in this section.

4.1. Let $p \in [2, m-1]$. Fix $w \in G(m,p,n)$ with $r = r(w)$. A subset E of $[r]$ is said to be *w-perfect* (respectively, *w-semi-perfect*) if E is $(C(w), m)$-perfect (respectively, $(C(w), m, p)$-semi-perfect) (see 1.5).

Lemma 4.2. *Let $w = [a_1, \dots, a_n \mid \sigma] \in G(m, p, n)$ be with $1 < p < m$. Let $s = t(i, j; a)$ be with $i < j$ in $[n]$ and $a \in \mathbb{Z}$. Then $r(ws) - v(ws) \leqslant r(w) - v(w) + 1$ (see* 1.7 *for the notation $v(w)$).*

Proof. Keep the notation in 1.7 and 4.1 for the element w.

(1) First assume $i, j \in I_k$ for some $k \in [r]$. We may assume $k = r$ by relabelling the I_h's if necessary. Then the partition of $[n]$ determined by ws is of the form $\{I'_1, \dots, I'_{r+1}\}$ with $I'_h = I_h$ for any $h \in [r-1]$ and $I'_r \cup I'_{r+1} = I_r$ by Lemma 1.1. Hence $r(ws) = r(w) + 1$. We must show $v(ws) \geqslant v(w)$. Take a partition $P = \{P_1, \dots, P_t, Q_1, \dots, Q_u\}$ of $[r]$ in $\Lambda(w; m, p)$ with $2t + u = v(w)$ such that $P_1, \dots, P_t$ are w-perfect and $Q_1, \dots, Q_u$ are w-semi-perfect. Assume $r \in P_l$ for some $l \in [t]$. We may assume $l = t$ by relabelling the P_h's if necessary. Then $P' = \{P_1, \dots, P_{t-1}, P_t \cup \{r+1\}, Q_1, \dots, Q_u\}$ is a partition of $[r+1]$ in $\Lambda(ws; m, p)$ with $P_1, \dots, P_{t-1}, P_t \cup \{r+1\}$ (ws)-perfect and $Q_1, \dots, Q_u$ (ws)-semi-perfect. This implies $v(P') \geqslant v(P)$ and hence $v(ws) \geqslant v(w)$. The proof is similar for the case where $r \in Q_l$ for some $l \in [u]$.

(2) Next assume $i \in I_k$ and $j \in I_l$ with $k \neq l$. We may assume $k = r-1$ and $l = r$ by relabelling the I_h's if necessary. Then the partition of $[n]$ determined by ws is of the form $\{I'_1, \dots, I'_{r-1}\}$ with $I'_h = I_h$ for any $h \in [r-2]$ and $I'_{r-1} = I_{r-1} \cup I_r$ by Lemma 1.1. Hence $r(ws) = r(w) - 1$. We must show $v(ws) \geqslant v(w) - 2$. Let $P = \{P_1, \dots, P_t, Q_1, \dots, Q_u\}$ be as in (1). First assume $r-1, r \in P_l$ for some $l \in [t]$. We may assume $l = t$ by relabelling the P_h's if necessary. Then $P' = \{P_1, \dots, P_{t-1}, P_t \setminus \{r\}, Q_1, \dots, Q_u\}$ is a partition of $[r-1]$ in $\Lambda(ws; m, p)$ with $P_1, \dots, P_{t-1}, P_t \setminus \{r\}$ (ws)-perfect and $Q_1, \dots, Q_u$ (ws)-semi-perfect. Next assume $r-1 \in P_k$ and $r \in P_l$ with $k \neq l$. We may assume $(k, l) = (t-1, t)$ by relabelling the P_h's if necessary. Then $P' = \{P_1, \dots, P_{t-2}, (P_{t-1} \cup P_t) \setminus \{r\}, Q_1, \dots, Q_u\}$ is a partition of $[r-1]$ in $\Lambda(ws; m, p)$ with $P_1, \dots, P_{t-2}, (P_{t-1} \cup P_t) \setminus \{r\}$ (ws)-perfect and $Q_1, \dots, Q_u$ (ws)-semi-perfect. The proof is similar for the case where $r-1, r \in Q_l$, or $r-1 \in Q_k$, $r \in Q_l$ with $k \neq l$. Finally assume $r-1 \in P_k$ and $r \in Q_l$ for some $k \in [t]$ and $l \in [u]$. We may assume $(k, l) = (t, u)$ by relabelling the P_h's and the Q_i's if necessary. Then $P' = \{P_1, \dots, P_{t-1}, Q_1, \dots, Q_{u-1}, (P_t \cup Q_u) \setminus \{r\}\}$ is a partition of $[r-1]$ in $\Lambda(ws; m, p)$ with $P_1, \dots, P_{t-1}$ (ws)-perfect and $Q_1, \dots, Q_{u-1}, (P_t \cup Q_u) \setminus \{r\}$ (ws)-semi-perfect. In either case, it is easily seen that $v(P') \geqslant v(P) - 2$. This implies the inequality $v(ws) \geqslant v(w) - 2$.

So our result follows by (1)–(2). □

Lemma 4.3. *Let $w = [a_1, \dots, a_n \mid \sigma] \in G(m, p, n)$ be with $p \in [2, m-1]$. Let $s = s(i; a)$, where $i \in [n]$ and $a \in \mathbb{Z}$ satisfy $a \equiv 0 \pmod p$ and $a \not\equiv 0 \pmod m$. Then $r(ws) - v(ws) \leqslant r(w) - v(w) + 1$.*

Proof. We have $r(ws) = r(w)$, denote this common value by r. Let $P = \{P_1, \dots, P_t, Q_1, \dots, Q_u\}$ be a partition of $[r]$ in $\Lambda(w; m, p)$ with $P_1, \dots, P_t$ w-perfect and $Q_1, \dots, Q_u$ w-semi-perfect. Keep the notation in 1.7 for w. Suppose $i \in I_k$ and $k \in P_j$ for some $k \in [r]$ and $j \in [t]$. We may assume $j = t$ by relabelling the P_h's if necessary. Then P is also a partition of $[r]$ in $\Lambda(ws; m, p)$ with $P_1, \dots, P_{t-1}$ (ws)-perfect and $Q_1, \dots, Q_u, P_t$ (ws)-semi-perfect. Next suppose $i \in I_k$ and $k \in Q_j$ for some $k \in [r]$ and $j \in [u]$. We may assume $j = u$ by relabelling the Q_h's if necessary. Then P is also a partition of $[r]$ in $\Lambda(ws; m, p)$ with $P_1, \dots, P_t$ (ws)-perfect, $Q_1, \dots, Q_{u-1}$ (ws)-semi-perfect, and Q_u is either (ws)-perfect or (ws)-semi-perfect. In either case, we have the inequality $v(ws) \geqslant v(w) - 1$. So $r(ws) - v(ws) \leqslant r(w) - v(w) + 1$. The result follows. □

Theorem 4.4. *Let $m, p, n \in \mathbb{P}$ be with $p \in [m]$ and $p \mid m$. Then*

$$l_T(w) = n + r(w) - v(w) \tag{4.4.1}$$

for any $w \in G(m, p, n)$.

Proof. Keep the notation in 1.7 for $w \in G(m, p, n)$. We know that $v(w) = t_0(w) + r(w)$ for any $w \in G(m, 1, n)$ and that $v(w) = 2t(w)$ for any $w \in G(m, m, n)$ (see 1.7). The result is just Theorems 2.1 and 3.1 when $p = 1, m$. So in the subsequent discussion, we always assume $p \in [2, m-1]$. Keep the notation in 1.7 for $w = [a_1, \ldots, a_n \mid \sigma]$. In particular, σ has the expression (1.7.1).

The result is true for $w = 1$ since $l_T(1) = 0$, $r(1) = n$ and $v(1) = 2n$. Now assume $w \neq 1$. Then $l := l_T(w) > 0$. There exist some reflections $s_1, \ldots, s_l$ in $G(m, p, n)$ with $ws_1s_2 \cdots s_l = 1$. By repeatedly applying Lemmas 4.2 and 4.3, we have

$$n + r(w) - v(w) \leqslant n + r(1) - v(1) + l = l_T(w). \tag{4.4.2}$$

It remains to show the inequality

$$l_T(w) \leqslant n + r(w) - v(w). \tag{4.4.3}$$

(i) Assume that the element w is in the subgroup $G(m, m, n)$ of $G(m, p, n)$ with $v(w) = 2$ (hence $t(w) = 1$). Then by Theorem 3.1, the element w can be written as a product of $n + r(w) - 2$ reflections of $G(m, p, n)$ in $G(m, m, n)$.

(ii) Assume $v(w) = 1$. Let $s = s(n; -c)$ be with $c = \sum_{j \in [r]} c_j$. Then s is a reflection in $G(m, p, n)$ and the element sw is in $G(m, m, n)$ with $r(sw) = r(w)$ and $t(sw) \geqslant 1$. By Theorem 3.1, we see that the element sw can be expressed as a product of at most $n + r(w) - 2$ reflections in $G(m, m, n)$. So w can be expressed as a product of at most $n + r(w) - 1$ reflections in $G(m, p, n)$.

(iii) Now assume that $P = \{P_1, \ldots, P_t, Q_1, \ldots, Q_u\}$ is a partition of $[r]$ for some $t, u \in \mathbb{N}$ with $2t + u = v(w)$ such that the P_j's are w-perfect and the Q_k's are w-semi-perfect. Let $n_j = \sum_{h \in P_j} |I_h|$, $n'_k = \sum_{h \in Q_k} |I_h|$, $p_j = |P_j|$ and $p'_k = |Q_k|$ for $j \in [t]$ and $k \in [u]$. Then $n = \sum_{j \in [t]} n_j + \sum_{k \in [u]} n'_k$ and $r(w) = \sum_{j \in [t]} p_j + \sum_{k \in [u]} p'_k$. For any $k \in [t]$, define $w_k = [a_{k1}, \ldots, a_{kn} \mid \sigma_k]$ by

$$a_{kj} = \begin{cases} a_j, & \text{if } j \in I_h \text{ for some } h \in P_k, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma_k = \prod_{h \in P_k} (i_{h1}, i_{h2}, \ldots, i_{hm_h}).$$

Also, for any $l \in [u]$, define $w'_l = [a_{l1}, \ldots, a_{ln} \mid \sigma'_l]$ by

$$a_{lj} = \begin{cases} a_j, & \text{if } j \in I_h \text{ for some } h \in Q_l, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sigma'_l = \prod_{h \in Q_l} (i_{h1}, i_{h2}, \ldots, i_{hm_h}).$$

By 1.4, we see that w_k (respectively, w'_l) can be identified with an element, say x_k (respectively, x'_l) in $G(m, m, n_k)$ (respectively, $G(m, p, n'_l)$) with $r(x_k) = p_k$ and $v(x_k) = 2$ (respectively, $r(x'_l) = p'_l$ and $v(x'_l) = 1$). By (ii), x_k (respectively, x'_l) can be expressed as a product of at most n_k+p_k-2 (respectively, $n'_l+p'_l-1$) reflections in $G(m,m,n_k)$ (respectively, $G(m,p,n'_l)$) and hence so does w_k (respectively, w'_l) in $G(m,p,n)$.

We have $w=\prod_{k\in[t]}w_k\cdot\prod_{l\in[u]}w'_l$. By (i)–(ii), the element w can be written as a product of at most

$$\sum_{j\in[t]}(n_j+p_j-2)+\sum_{h\in[u]}\bigl(n'_h+p'_h-1\bigr)=n+r(w)-2t-u=n+r(w)-v(w)$$

reflections in $G(m,p,n)$. This implies the inequality (4.4.3). The result is proved. □

Example 4.5. Let $w=[2,4,8,5,2,3,2,7\mid(18)(23)(457)(6)]\in G(12,3,8)$. Then $r(w)=4$. We have $C(w)=[\![c_1,c_2,c_3,c_4]\!]=[\![9,12,9,3]\!]=[\![12]\!]\cup[\![9,3]\!]\cup[\![9]\!]$. So $v(w)=2\cdot2+1=5$. Hence $l_T(w)=8+4-5=7$.

5. Some properties of the function $l_T(w)$

In applying Theorem 4.4 to calculate $l_T(w)$, a problem is how to get the value $v(w)$ in the case of $w\in G(m,p,n)$ with $p\in[2,m-1]$. Let $\Lambda_0(w;m,p)$ be the subset of $\Lambda(w;m,p)$ consisting of all the elements P with $2t(P)+u(P)=v(w)$. Then the above problem is amount to picking out an element of $\Lambda_0(w;m,p)$ from the set $\Lambda(w;m,p)$.

Keep the notation in 1.7 for the element w. The following are some properties for the elements in $\Lambda_0(w;m,p)$.

Lemma 5.1. *Take $P=\{P_1,\dots,P_t,Q_1,\dots,Q_u\}\in\Lambda_0(w;m,p)$, where all the P_j's are w-perfect and all the Q_k's are w-semi-perfect. Then the following conditions hold*:

(1) *No proper subset of P_j and Q_k is w-perfect.*
(2) *No proper subset of Q_k is w-semi-perfect.*
(3) *If $c_j\equiv 0 \pmod m$ for some $j\in[r]$, then $P_l=\{j\}$ for some $l\in[t]$.*
By taking a suitable choice, we can find some $P\in\Lambda_0(w;m,p)$ to satisfy one more condition as follows:
(4) *No proper subset of P_j is w-semi-perfect. In particular, $\{j\}$ is a block of P whenever $c_j\equiv 0 \pmod p$.*

Proof. (1) If some P_j (respectively, Q_k) contains a proper w-perfect subset, say P_{j1} (respectively, Q_{k1}), let $P_{j2}=P_j\setminus P_{j1}$ (respectively, $Q_{k2}=Q_k\setminus Q_{k1}$), then P_{j2} is w-perfect (respectively, Q_{k2} is w-semi-perfect). Then

$$P'=\{P_1,\dots,\widehat{P_j},\dots,P_t,Q_1,\dots,Q_u,P_{j1},P_{j2}\}$$
$$\bigl(\text{respectively,}\quad P''=\{P_1,\dots,P_t,Q_1,\dots,\widehat{Q_k},\dots,Q_u,Q_{k1},Q_{k2}\}\bigr)$$

is in $\Lambda(w;m,p)$ with

$$2t(P')+u(P')=2t(P)+u(P)+2=v(w)+2>v(w)$$
$$\bigl(\text{respectively,}\quad 2t(P'')+u(P'')=2t(P)+u(P)+2=2v(w)+2>v(w)\bigr),$$

contradicting the choice of P.

(2) If some Q_k contains a proper w-semi-perfect subset, say Q_{k1}, let $Q_{k2} = Q_k \setminus Q_{k1}$, then Q_{k2} is either w-perfect or w-semi-perfect. Hence

$$P' = \{P_1, \ldots, P_t, Q_1, \ldots, \widehat{Q_k}, \ldots, Q_u, Q_{k1}, Q_{k2}\}$$

is in $\Lambda(w; m, p)$ with $2t(P') + u(P') \geqslant 2t(P) + u(P) + 1 = v(w) + 1 > v(w)$, again contradicting the choice of P.

(3) Under the assumption of (3), if $\{j\} \subsetneqq P_l$ for some $l \in [t]$, let $P_{l1} = P_l \setminus \{j\}$, then P_{l1} is also w-perfect; if $\{j\} \subsetneqq Q_k$ for some $k \in [u]$, let $Q_{k1} = Q_k \setminus \{j\}$, then Q_{k1} is w-semi-perfect. Let

$$P' = \begin{cases} \{P_1, \ldots, \widehat{P_l}, \ldots, P_t, Q_1, \ldots, Q_u, \{j\}, P_{l1}\}, & \text{if } \{j\} \subsetneqq P_l \text{ for some } l \in [t], \\ \{P_1, \ldots, P_t, Q_1, \ldots, \widehat{Q_k}, \ldots, Q_u, \{j\}, Q_{k1}\}, & \text{if } \{j\} \subsetneqq Q_k \text{ for some } k \in [u]. \end{cases}$$

Then $2t(P') + u(P') = 2(t(P) + 1) + u(P) = v(w) + 2 > v(w)$, contradicting the choice of P.

(4) If P_j contains a proper w-semi-perfect subset, say P_{j1}, let $P_{j2} = P_j \setminus P_{j1}$, then P_{j2} is also w-semi-perfect. Hence $P' = \{P_1, \ldots, \widehat{P_j}, \ldots, P_t, Q_1, \ldots, Q_u, P_{j1}, P_{j2}\}$ is in $\Lambda(w; m, p)$ with $2t(P') + u(P') = 2t(P) + u(P) = v(w)$. Denote by $n(P)$ the number of all the w-perfect parts P_j each of which contains a proper w-semi-perfect subset. Replace P by P' and apply induction on the number $n(P) \geqslant 0$, we can eventually get some P'' in $\Lambda(w; m, p)$ with $n(P'') = 0$, as required. $\square$

Define $t_1(w) = \max\{t(P) \mid P \in \Lambda(w; m, p)\}$ and $u_1(w) = \max\{u(P) \mid P \in \Lambda(w; m, p)\}$ for any $w \in G(m, p, n)$.

We have $t_1(w) = t(w)$ for $w \in G(m, m, n)$. For any $w \in G(m, p, n)$, the inequalities $r(w) \geqslant t_1(w) \geqslant t_0(w)$ hold in general; the equality $r(w) = t_1(w)$ holds if and only if $w \in G(m, m, n)$ and $t_1(w) = t_0(w)$.

Lemma 5.2. *$l_T(w) \geqslant n - t_1(w)$ for any $w \in G(m, p, n)$. The equality holds if and only if there exists some $P = (P_1, P_2, \ldots, P_t) \in \Lambda(w; m, p)$ with $t_1(w) = t(P)$ and $|P_i| = 1$ for any $i \in [t]$ (hence $t = r(w)$, $t_1(w) = t_0(w)$ and $P \in \Lambda_0(w; m, p)$, see* 1.5*).*

Proof. By the definition of $v(w)$ in 1.7, we see that the inequality $r(w) \geqslant t(P) + u(P)$ holds for any $P \in \Lambda(w; m, p)$. Hence by Theorem 4.4, we have

$$\begin{aligned} l_T(w) &= n + \min\bigl\{r(w) - 2t(P) - u(P) \bigm| P \in \Lambda(w; m, p)\bigr\} \\ &\geqslant n - \max\bigl\{t(P) \bigm| P \in \Lambda(w; m, p)\bigr\} = n - t_1(w). \end{aligned}$$

Hence $l_T(w) = n - t_1(w)$ if and only if $r(w) = t(P) + u(P)$ and $t_1(w) = t(P)$ for some $P \in \Lambda(w; m, p)$ by Lemma 5.1. So the proof is completed. $\square$

By Lemma 5.2, one can deduce the following results.

Proposition 5.3. *Let $w \in G(m, p, n)$.*

(1) *w is a reflection if and only if $t_1(w) = n - 1$.*
(2) *$l_T(w) = \operatorname{codim}_V V^w$ (see* 0.2*) if and only if there exists some $P = (P_1, P_2, \ldots, P_t) \in \Lambda_0(w; m, p)$ with $|P_i| \leqslant 2$ for any $i \in [t]$ such that $|P_i| = 2$ only if P_i is w-perfect.*

(3) *$l_T(w) < n$ only if $t_1(w) \geqslant 1$, or equivalently, if $t_1(w) = 0$ then $l_T(w) \geqslant n$.*
(4) *If $t_1(w) \geqslant \frac{r(w)}{2}$, then $l_T(w) \leqslant n$.*
(5) *If $t_1(w) = r(w)$, then $l_T(w) = n - r(w)$.*
(6) *If $u_1(w) = r(w)$, then $l_T(w) = n$.*
(7) *$l_T(w) \leqslant 2n - 1$ for any $w \in G(m, p, n)$.*
(8) *$l_T(w) = 2n - 1$ if and only if $r(w) = n$, $t_1(w) = 0$ and $u_1(w) = 1$. When the equivalent conditions hold, the element w is a diagonal matrix such that the product of any $k < n$ diagonal entries is not an (m/p)th root of unity.*

Proof. (1) The implication "⇒" follows by the definition of a reflection (see the conditions (1)–(2) in the Introduction). For the implication "⇐," we have the inequalities $n \geqslant r(w) \geqslant t_1(w) = n - 1$. If $r(w) = n$, then w is diagonal with exactly $n - 1$ diagonal entries equal to 1; if $r(w) = n - 1$, then w has $n - 2$ diagonal entries equal to 1 and has a 2×2 monomial minor which is non-diagonal with the product of non-zero entries equal to 1. Hence w is a reflection in either case.

(2) By Theorem 4.4, we see that the equation $l_T(w) = \operatorname{codim}_V V^w$ if and only if there exists some $P \in \Lambda_0(w; m, p)$ with $r(w) + t_0(w) = 2t(P) + u(P)$. Then our result follows easily by Lemma 5.1(3).

(3) is an easy consequence of Lemma 5.2.

(4) $l_T(w) = n + r(w) - v(w) \leqslant n + r(w) - 2t_1(w) \leqslant n$ by Theorem 4.4, the fact $v(w) \geqslant 2t_1(w)$ and the assumption $t_1(w) \geqslant \frac{r(w)}{2}$.

(5) By the discussion in the paragraph preceding Lemma 5.2, we have $w \in G(m, m, n)$ and $t_1(w) = t(w)$. So the result follows by Theorem 3.1.

(6) Under the assumption of $u_1(w) = r(w)$, we see that for each $P \in \Lambda(w; m, p)$, any w-perfect block of P contains at least two elements of $[r(w)]$. Then $t(P) \leqslant \frac{1}{2}(r(w) - u(P))$, i.e., $2t(P) + u(P) \leqslant r(w)$. This implies $l_T(w) \geqslant n$ by Theorem 4.4. On the other hand, there exists some $P \in \Lambda(w; m, p)$ with $u(P) = u_1(w) = r(w)$. Hence $l_T(w) = n + r(w) - v(w) \leqslant n + r(w) - u(P) = n$ again by Theorem 4.4. So our result follows.

(7) This follows by Theorem 4.4 and the facts that $n \geqslant r(w)$ and $v(w) > 0$.

(8) We see by Theorem 4.4 and (7) that $l_T(w) = 2n - 1$ if and only if $r(w) = n$ and $v(w) = 1$, while $v(w) = 1$ if and only if $u_1(w) = 1$ and $t_1(w) = 0$. This proves that $l_T(w) = 2n - 1$ if and only if $r(w) = n$, $t_1(w) = 0$ and $u_1(w) = 1$. Then the remaining assertion follows immediately. □

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